

UNIQUENESS, UNIVERSALITY, AND HOMOGENEITY OF THE NONCOMMUTATIVE GURARIJ SPACE

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ABSTRACT. We realize the noncommutative Gurarij space \mathbb{NG} defined by Oikhberg as the Fraïssé limit of the class of finite-dimensional 1-exact operator spaces. As a consequence we deduce that the noncommutative Gurarij space is unique up to completely isometric isomorphism, homogeneous, and universal among separable 1-exact operator spaces. Moreover we show that \mathbb{NG} is isometrically isomorphic to the Gurarij Banach space.

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1. INTRODUCTION

The *Gurarij space* \mathbb{G} is a Banach space first constructed by Gurarij in [14]. It has the following universal property: whenever $X \subset Y$ are finite-dimensional Banach spaces, $\phi : X \rightarrow \mathbb{G}$ is a linear isometry, and $\varepsilon > 0$, there is an injective linear map $\psi : Y \rightarrow \mathbb{G}$ extending ϕ such that $\|\psi\| \|\psi^{-1}\| < 1 + \varepsilon$. The uniqueness of such an object was proved by Lusky [23]. A short proof was later provided by Kubis-Solecki [19]. The Gurarij space was first realized as a Fraïssé limit by Ben Yaacov in [1].

Fraïssé theory is a subject at the border between model theory and combinatorics originating from the seminar work of Fraïssé [11]. Broadly speaking, Fraïssé theory studies homogeneous structures and ways to construct them. In the discrete setting Fraïssé established in [11] a correspondence between countable homogeneous structures and what are now called Fraïssé classes.

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Let the *age* of a countable structure \mathbb{S} be the collection of finitely generated substructures of \mathbb{S} . Any Fraïssé class is the age of a countable homogeneous structure. Conversely from any Fraïssé class one can build a countable homogeneous structure that has the given class as its age. Moreover such a structure is uniquely determined up to isomorphism by this property.

This correspondence has been recently generalized in [1] by Ben Yaacov from the purely discrete setting to the setting where metric structures are considered; see also [31]. The main results of discrete Fraïssé theory are recovered in this more general framework. In particular any Fraïssé class of metric structures is the age of a separable homogeneous structure, which is unique up to isometric isomorphism. An alternative categorical-theoretic approach to Fraïssé limits in the metric setting has been developed by Kubiś [18].

The Gurarij space is the limit of the Fraïssé class of finite-dimensional Banach spaces. This has been showed in [1] building on previous work of Henson. In particular this has yielded an alternative proof of the uniqueness of the Gurarij space up to isometric isomorphism. Other naturally occurring examples of Fraïssé limits are the Urysohn universal metric space [24], the hyperfinite II_1 factor and the infinite type UHF C^* -algebras [5].

In this paper we consider a noncommutative analog of the Gurarij space introduced by Oikhberg in [25] within the framework of operator spaces. Operator spaces can be regarded as noncommutative Banach spaces. In fact Banach spaces can be concretely defined as closed subspaces of $C(K)$ spaces, where K is a compact Hausdorff space. These are precisely the *abelian* unital C^* -algebras. Replacing abelian C^* -algebras with arbitrary C^* -algebras or—equivalently—the algebra $B(H)$ of bounded linear operators on some Hilbert space H provides the notion of an *operator space*.

An operator space $X \subset B(H)$ is endowed with matricial norms on the algebraic tensor product $M_n \otimes X$ obtained by the inclusion $M_n \otimes X \subset M_n \otimes B(H) \cong B(H \oplus \cdots \oplus H)$. A linear operator ϕ between operator spaces is completely bounded with norm at most M if all its amplifications $\text{id}_{M_n} \otimes \phi$ are bounded with norm at most M . The notion of complete isometry is defined similarly. Operator spaces form then a category with completely bounded (or completely isometric) linear maps as morphisms. Any Banach space X has a canonical operator space structure induced by the inclusion $X \subset C(\text{Ball}(X^*))$ where $\text{Ball}(X^*)$ is the unit ball of the dual of X . However in this case the matricial norms do not provide any new information, and any linear map ϕ between Banach spaces is automatically completely bounded with same norm. For more general operator spaces it is far from being true that any bounded linear map is completely bounded. The matricial norms play in this case a crucial role.

According to [25] an operator space is *noncommutative Gurarij* if it satisfies the same universal property of the Gurarij Banach space, where finite-dimensional Banach spaces are replaced with arbitrary *finite-dimensional*

1-exact operator spaces, and the operator norm is replaced by the *completely bounded norm*. The restriction to 1-exact spaces is natural since a famous result of Junge and Pisier asserts that there is no separable operator space containing all the finite-dimensional operator spaces as subspaces [16, Theorem 2.3]; see also [29, Chapter 21]. Proposition 3.1 of [25] shows that any two noncommutative Gurarij are *approximately* completely isometrically isomorphic. Moreover [25, Theorem 1.1] shows that separable $\mathcal{OL}_{\infty,1+}$ spaces—as defined in [7, 15]—can be completely isometrically embedded in some noncommutative Gurarij space as completely contractively complemented subspaces. Since $\mathcal{OL}_{\infty,1+}$ spaces are the noncommutative analog of $\mathcal{L}_{\infty,1+}$ Banach spaces, this result is the analog of the classical fact that $\mathcal{L}_{\infty,1+}$ Banach spaces can be isometrically embedded in the Gurarij Banach space as contractively complemented subspaces; see [17, Theorem 3.21] and [20, Theorem 1]. In view of [6, Theorem 4.7], Oikhberg’s result implies that every separable 1-exact operator space can be completely isometrically embedded in some noncommutative Gurarij space.

The main result of this paper is that the noncommutative Gurarij space can be realized as the limit of the Fraïssé class of finite-dimensional 1-exact operator spaces. We deduce as a consequence that the noncommutative Gurarij space—which we denote by NG —is unique up to complete isometry, universal among separable 1-exact operator spaces, and moreover satisfies the following homogeneity property: for any finite-dimensional subspace $X \subset \mathsf{NG}$, any complete isometry $\phi : X \rightarrow \mathsf{NG}$, and any $\varepsilon > 0$ there is a surjective linear complete isometry $\psi : \mathsf{NG} \rightarrow \mathsf{NG}$ such that $\psi|_X - \phi$ has completely bounded norm at most ε .

This rest of this paper is divided into three sections. Section 2 contains some background material on Fraïssé theory and operator spaces. We follow the presentation of Fraïssé theory for metric structures as introduced by Ben Yaacov in [1]. Similarly as [24] we adopt the slightly less general point of view—sufficient for our purposes—where one considers only structures where the interpretation of function and relation symbols are Lipschitz with a constant that does not depend on the structure. The material on operator spaces is standard and can be found for example in the monographs [8, 27, 29]. The topic of M_n -spaces is perhaps less well known and can be found in Lehrer’s PhD thesis [21] as well as in [25, 26].

In Section 3 we show that the class of finite-dimensional M_n -spaces is a Fraïssé class. This can be seen as a first step towards proving that the class of finite-dimensional 1-exact operator spaces is a Fraïssé class. Any M_n -space can be canonically endowed with a compatible operator space structure. Therefore in principle it is possible to rephrase all the arguments and results in terms of operator spaces. Nonetheless we find it more convenient and enlightening to deal with M_n -space. This allows one to recognize and use the analogy with the Banach space case.

Finally Section 4 contains the proof of the main result, asserting that the class of finite-dimensional 1-exact operator spaces is a Fraïssé class. Its limit is then identified as the noncommutative Gurarij space.

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2. BACKGROUND MATERIAL

2.1. Approximate isometries. Suppose that A, B are complete metric spaces. An *approximate isometry* from A to B is a map $\psi : A \times B \rightarrow [0, +\infty]$ satisfying the following:

$$|\psi(a, b) - \psi(a', b)| \leq d(a, a') \leq \psi(a, b) + \psi(a', b)$$

and

$$|\psi(a, b) - \psi(a, b')| \leq d(b, b') \leq \psi(a, b) + \psi(a, b')$$

for every $a, a' \in A$ and $b, b' \in B$. If ψ is an approximate isometry from A to B , then we write $\psi : A \rightsquigarrow B$. The set of all approximate isometries from A to B is denoted by $\text{Apx}(A, B)$. This is a compact space endowed with the product topology from $[0, +\infty]^{A \times B}$. A *partial isometry* f from A to B is an isometry from a subset $\text{dom}(f)$ of A to B .

Remark 2.1. Any partial isometry f will be identified with the approximate isometry ψ_f given by the distance function from the graph of f .

Explicitly ψ_f is defined by

$$\psi_f(a, b) = \inf_{z \in \text{dom}(f)} (d(a, z) + d(f(z), b)).$$

If $\psi : A \rightsquigarrow B$ one can consider its *pseudo-inverse* $\psi^* : B \rightsquigarrow A$ defined by $\psi^*(b, a) = \psi(a, b)$. Moreover one can take *composition* of approximate isometries $\psi : A \rightsquigarrow B$ and $\phi : B \rightsquigarrow C$ by setting

$$(\phi\psi)(a, c) = \inf_{b \in B} (\psi(a, b) + \phi(b, c)).$$

These definitions are consistent with composition and inversion of partial isometries when regarded as approximate isometries.

If $A_0 \subset A$, $B_0 \subset B$, and $\psi : A \rightsquigarrow B$ then one can define the *restriction* $\psi|_{A_0 \times B_0} = j^* \psi i : A_0 \rightsquigarrow B_0$ where i and j are the inclusion maps of A_0 into A and B_0 into B . Conversely if $\phi : A_0 \rightsquigarrow B_0$ then one can consider its *trivial extension* $j \phi i^* : A \rightsquigarrow B$. This allows one to regard $\text{Apx}(A_0, B_0)$ as a subset of $\text{Apx}(A, B)$ by identifying an approximate isometry with its trivial extension.

For approximate isometries $\phi, \psi : A \rightsquigarrow B$ we say that ϕ *refines* ψ and ψ *coarsens* ϕ —written $\phi \leq \psi$ —if $\phi(a, b) \leq \psi(a, b)$ for every $a \in A$ and $b \in B$. The set of approximate isometries that refine ψ is denoted by $\text{Apx}^{\leq \psi}(A, B)$. The interior of $\text{Apx}^{\leq \psi}(A, B)$ is denoted by $\text{Apx}^{< \psi}(A, B)$. The *closure under coarsening* \mathcal{A}^\uparrow of a set $\mathcal{A} \subset \text{Apx}(A, B)$ is the collection of $\phi \in \text{Apx}(A, B)$ that coarsen some element of \mathcal{A} .

2.2. Languages and structures. A *language* \mathcal{L} is given by sets of predicate symbols and of function symbols. Every symbol has two natural numbers attached: its arity and its Lipschitz constant. An \mathcal{L} -*structure* \mathfrak{A} is given by

- a complete metric space A ,
- a c_B -Lipschitz function $B^\mathfrak{A} : A^{n_B} \rightarrow \mathbb{R}$ for every predicate symbol B , where c_B is the Lipschitz constant of B and n_B is the arity of B , and
- a c_f -Lipschitz function $f^\mathfrak{A} : A^{n_f} \rightarrow A$ for every function symbol f , where c_f is the Lipschitz constant of f and n_f is the arity of f .

Here and in the following we assume the power A^n to be endowed with the max metric $d(\bar{a}, \bar{b}) = \max_i d(a_i, b_i)$. An *embedding* of \mathcal{L} -structures $\phi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a function that commutes with the interpretation of all the predicate and function symbols. An *isomorphism* is a surjective embedding. An *automorphism* of \mathfrak{A} is an isomorphism from \mathfrak{A} to \mathfrak{A} . If \bar{a} is a finite tuple in \mathfrak{A} then $\langle \bar{a} \rangle$ denotes the smallest substructure of \mathfrak{A} containing \bar{a} . A *partial isomorphism* $\phi : \mathfrak{A} \dashrightarrow \mathfrak{B}$ is an embedding from $\langle \bar{a} \rangle$ to \mathfrak{B} for some finite tuple \bar{a} in \mathfrak{A} . An \mathcal{L} -structure \mathfrak{A} is *finitely generated* if $\mathfrak{A} = \langle \bar{a} \rangle$ for some finite tuple \bar{a} in \mathfrak{A} .

We will assume that the language \mathcal{L} contains a distinguished binary predicate symbol to be interpreted as the metric. In particular this ensures that all the embeddings and (partial) isomorphisms are (partial) isometries. Therefore consistently with the convention from Remark 2.1 partial isomorphisms will be regarded as approximate isometries.

Definition 2.2. Suppose that \mathcal{C} is a class of finitely-generated \mathcal{L} -structure. We say that \mathcal{C} satisfies

- the *hereditary property* (HP) if $\langle \bar{a} \rangle \in \mathcal{C}$ for every $\mathfrak{A} \in \mathcal{C}$ and finite tuple $\bar{a} \in \mathfrak{A}$,
- the *joint embedding property* (JEP) if for any $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ there is $\mathfrak{C} \in \mathcal{C}$ and embeddings $\phi : \mathfrak{A} \rightarrow \mathfrak{C}$ and $\psi : \mathfrak{B} \rightarrow \mathfrak{C}$,
- the *near amalgamation property* (NAP) if, whenever $\mathfrak{A} \subset \mathfrak{B}_0$ and \mathfrak{B}_1 are elements of \mathcal{C} , $\phi : \mathfrak{A} \rightarrow \mathfrak{B}_1$ is an embedding, \bar{a} is a finite tuple in \mathfrak{A} , and $\varepsilon > 0$, there exists $\mathfrak{C} \in \mathcal{C}$ and embeddings $\psi_0 : \mathfrak{B}_0 \rightarrow \mathfrak{C}$ and $\psi_1 : \mathfrak{B}_1 \rightarrow \mathfrak{C}$ such that

$$d(\psi_0(\bar{a}), (\psi_1 \circ \phi)(\bar{a})) \leq \varepsilon.$$

- the *amalgamation property* (AP) if it satisfies (NAP) even when one takes $\varepsilon = 0$.

2.3. Fraïssé classes and limits. Suppose in the following that \mathcal{C} is a class of finitely generated \mathcal{L} -structures satisfying (HP), (JEP), and (NAP).

Definition 2.3. A \mathcal{C} -structure is an \mathcal{L} -structure \mathfrak{A} such that $\langle \bar{a} \rangle \in \mathcal{C}$ for every finite tuple \bar{a} in \mathfrak{A} .

Let \mathfrak{A} and \mathfrak{B} be \mathcal{C} -structures. Define $\text{Apx}_{1,\mathcal{C}}(\mathfrak{A}, \mathfrak{B}) \subset \text{Apx}(A, B)$ to be the set of all partial isomorphisms from \mathfrak{A} to \mathfrak{B} . Define $\text{Apx}_{2,\mathcal{C}}(\mathfrak{A}, \mathfrak{B})$ to be the set of approximate isometries $\phi : A \rightsquigarrow B$ of the form

$$\phi = g^* f$$

where $f \in \text{Apx}_{1,\mathcal{C}}(\mathfrak{A}, \mathfrak{C})$ and $g \in \text{Apx}_{1,\mathcal{C}}(\mathfrak{B}, \mathfrak{C})$ for some \mathcal{C} -structure \mathfrak{C} .

Finally set $\text{Apx}_{\mathcal{C}}(\mathfrak{A}, \mathfrak{B}) \subset \text{Apx}(A, B)$ to be $\overline{\text{Apx}_{2,\mathcal{C}}(\mathfrak{A}, \mathfrak{B})}^{\uparrow}$. Elements of $\text{Apx}_{\mathcal{C}}(\mathfrak{A}, \mathfrak{B})$ are called (\mathcal{C} -intrinsic) *approximate morphism*. A (\mathcal{C} -intrinsic) *strictly approximate morphism* from \mathfrak{A} to \mathfrak{B} is an approximate morphism ϕ such that the interior $\text{Apx}_{\mathcal{C}}^{<\phi}(\mathfrak{A}, \mathfrak{B})$ of $\text{Apx}_{\mathcal{C}}^{<\phi}(\mathfrak{A}, \mathfrak{B})$ is nonempty. The set of strictly approximate morphisms from \mathfrak{A} to \mathfrak{B} is denoted by $\text{Stx}_{\mathcal{C}}(\mathfrak{A}, \mathfrak{B})$.

Fix $k \in \mathbb{N}$ and denote by $\mathcal{C}(k)$ the set of pairs (\bar{a}, \mathfrak{A}) where $\mathfrak{A} \in \mathcal{C}$ and \bar{a} is a finite tuple in \mathfrak{A} such that $\mathfrak{A} = \langle \bar{a} \rangle$. Two such pairs (\bar{a}, \mathfrak{A}) and (\bar{b}, \mathfrak{B}) are identified if there is an isomorphism $\phi : \mathfrak{A} \rightarrow \mathfrak{B}$ such that $\phi(\bar{a}) = \bar{b}$. By abuse of notation we will denote (\bar{a}, \mathfrak{A}) simply by \bar{a} .

Definition 2.4. The Fraïssé metric $d_{\mathcal{C}}$ on $\mathcal{C}(k)$ is defined by

$$d_{\mathcal{C}}(\bar{a}, \bar{b}) = \inf_{\phi} \max_i \phi(a_i, b_i)$$

where ϕ ranges in $\text{Apx}_{\mathcal{C}}(\langle \bar{a} \rangle, \langle \bar{b} \rangle)$ or, equivalently, in $\text{Stx}_{\mathcal{C}}(\langle \bar{a} \rangle, \langle \bar{b} \rangle)$.

Such a metric can be equivalently described in terms of embeddings:

$$d_{\mathcal{C}}(\bar{a}, \bar{b}) = \inf_{f,g} d(f(\bar{a}), g(\bar{b}))$$

where f, g range over all the embeddings of $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ into a third structure $\mathfrak{C} \in \mathcal{C}$.

Definition 2.5. Suppose that \mathcal{C} is a class of finitely-generated \mathcal{L} -structures satisfying (HP), (JEP), and (NAP) from Definition 2.2. We say that \mathcal{C} is a *Fraïssé class* if the metric space $(\mathcal{C}(k), d_{\mathcal{C}})$ is complete and separable for every $k \in \mathbb{N}$.

Remark 2.6. In [1, Definition 2.12] a Fraïssé class is moreover required to satisfy the Continuity Property. Such a property is automatically satisfied in our more restrictive setting, where we assume that the interpretation of any symbol from \mathcal{L} is a Lipschitz function with Lipschitz constant that does not depend from the structure.

Definition 2.7. Suppose that \mathcal{C} is a Fraïssé class. A limit of \mathcal{C} is a separable \mathcal{C} -structure \mathfrak{M} satisfying the following property: For every $\mathfrak{A} \in \mathcal{C}$, finite tuple \bar{a} in \mathfrak{A} , embedding $\phi : \langle \bar{a} \rangle \rightarrow \mathfrak{M}$, and $\varepsilon > 0$ there is an embedding $\psi : \mathfrak{A} \rightarrow \mathfrak{M}$ such that $d(\psi(\bar{a}), \phi(\bar{a})) < \varepsilon$.

The definition given above is equivalent to [1, Definition 2.14] in view of [1, Corollary 2.20].

Definition 2.8. An \mathcal{L} -structure \mathfrak{M} is *homogeneous* if for every finite tuple \bar{a} in \mathfrak{A} , embedding $\phi : \langle \bar{a} \rangle \rightarrow \mathfrak{A}$, and $\varepsilon > 0$, there is an automorphism ψ of \mathfrak{M} such that $d(\phi(\bar{a}), \psi(\bar{a})) < \varepsilon$.

The following Theorem is a combination of the main results from [1].

Theorem 2.9 (Ben Yaacov). Suppose that \mathcal{C} is a Fraïssé class. Then \mathcal{C} has a limit \mathfrak{M} . If \mathfrak{M}' is another limit of \mathcal{C} then \mathfrak{M} and \mathfrak{M}' are isomorphic as \mathcal{L} -structures. Moreover \mathfrak{M} is homogeneous and contains any separable \mathcal{C} -structure as a substructure.

Remark 2.10. Suppose that \mathcal{C} is a Fraïssé class. Assume that \mathcal{A} is a class of separable \mathcal{C} -structure with the following properties:

- \mathcal{A} is closed under isomorphism and countable direct limits, and
- every separable \mathcal{C} -structure embeds into an element of \mathcal{A} .

It is clear from the proof of [1, Lemma 2.17] that the Fraïssé limit \mathfrak{M} of \mathcal{C} belongs to \mathcal{A} .

2.4. Operator spaces. An operator space is a closed subspace of $B(H)$. Here and in the following we denote by M_n the algebra of $n \times n$ complex matrices. If X is a complex vector space, then we denote by $M_n \otimes X$ the algebraic tensor product. Observe that this can be canonically identified with the space $M_n(X)$ of $n \times n$ matrices with entries from X . If $X \subset B(H)$ is an operator space, then $M_n(X)$ is naturally endowed with a norm given by the inclusion $M_n(X) \subset B(H \oplus \cdots \oplus H)$. If $\phi : X \rightarrow Y$ is a linear map between operator spaces, then its n -th amplification is the linear map $id_{M_n} \otimes \phi : M_n \otimes X \rightarrow M_n \otimes Y$. Under the identification of $M_n \otimes X$ with $M_n(X)$ and $M_n \otimes Y$ with $M_n(Y)$, the map $id_{M_n} \otimes \phi$ is defined by

$$(id_{M_n} \otimes \phi)[x_{ij}] = [\phi(x_{ij})].$$

We say that ϕ is completely bounded if

$$\sup_n \|id_{M_n} \otimes \phi\| < +\infty.$$

In such case we define its completely bounded norm

$$\|\phi\|_{cb} = \sup_n \|id_{M_n} \otimes \phi\|.$$

A linear map ϕ is a complete contraction if $id_{M_n} \otimes \phi$ is a contraction for every $n \in \mathbb{N}$. It is a complete isometry if $id_{M_n} \otimes \phi$ is an isometry for every $n \in \mathbb{N}$. Finally it is a completely isometric isomorphism if $id_{M_n} \otimes \phi$ is an isometric isomorphism for every $n \in \mathbb{N}$.

Operator spaces admit an abstract characterization due to Ruan [30]. A vector space X is matrix-normed if for every $n \in \mathbb{N}$ the space $M_n(X)$ is endowed with a norm such that whenever $\alpha \in M_{k,n}$, $x \in M_n(X)$, and $\beta \in M_{n,k}$

$$\|\alpha.x.\beta\|_k \leq \|\alpha\| \|x\| \|\beta\|$$

where $\alpha.x.\beta$ denotes the matrix product, and $\|\alpha\|, \|\beta\|$ are the norms of α and β regarded as operators between finite-dimensional Hilbert spaces. A matrix-normed vector space is L^∞ -matrix-normed provided that

$$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\| = \max \{ \|x\|, \|y\| \}$$

for $x \in M_n(X)$ and $y \in M_m(X)$. Every operator space $X \subset B(H)$ is canonically an L^∞ -matrix-normed space. Ruan's theorem asserts that, conversely, any L^∞ -matrix-normed space is completely isometrically isomorphic to an operator space [27, Theorem 13.4].

Equivalently one can think of an operator system X as a structure on $\mathcal{K}_0 \otimes X$; see [29, Section 2.2]. Suppose that H is the separable Hilbert space with a fixed orthonormal basis $(e_k)_{k \in \mathbb{N}}$. Let P_n be the orthogonal projection of $\text{span} \{e_1, \dots, e_n\}$ for $n \in \mathbb{N}$. We can identify M_n with the subspace of $A \in B(H)$ such that $AP_n = P_n A = A$. The direct union $\mathcal{K}_0 = \bigcup_n M_n$ is a subspace of $B(H)$. We can identify $\bigcup_n M_n(X)$ with $\mathcal{K}_0[X] \cong \mathcal{K}_0 \otimes X$. Then $\mathcal{K}_0[X]$ is a complex vector space with a natural structure of \mathcal{K}_0 -bimodule. In formulas if $\alpha, \beta \in \mathcal{K}_0$ and $x = \sum_i \gamma_i \otimes y_i \in \mathcal{K}_0[X]$, then

$$\alpha.x.\beta = \sum_i \alpha \gamma_i \beta \otimes y_i \in \mathcal{K}_0[X].$$

In this framework one can reformulate Ruan's axioms as follows; see [29, Page 35]. If $\alpha_i, \beta_i \in \mathcal{K}_0$ and $x_i \in \mathcal{K}_0[X]$ are finite sequences then

$$\left\| \sum_i \alpha_i.x_i.\beta_i \right\| \leq \left\| \sum_i \alpha_i^* \alpha_i \right\|^{\frac{1}{2}} \max_i \|x_i\| \left\| \sum_i \beta_i \beta_i^* \right\|^{\frac{1}{2}}.$$

The metric on $\mathcal{K}_0[X]$ is not complete. Nonetheless one can pass to the completion $\mathcal{K} \overline{\otimes} X$ and extend all the operations. (Here \mathcal{K} is the closure of \mathcal{K}_0 inside $B(H)$, i.e. the ideal of compact operators.)

The abstract characterization of operator spaces mentioned above allows one to regard operator spaces as \mathcal{L}_{OS} -structures for a suitable language \mathcal{L}_{OS} . Denote by

$$\mathcal{K}_0(\mathbb{Q}(i)) = \bigcup_{n \in \mathbb{N}} M_n(\mathbb{Q}(i))$$

the space of matrices with coefficients in the field of Gauss rationals $\mathbb{Q}(i)$. Then \mathcal{L}_{OS} contains, in addition to the special symbol d for the metric, a symbol $+$ for the addition in $\mathcal{K} \overline{\otimes} X$, a constant 0 for the zero vector in $\mathcal{K} \overline{\otimes} X$, function symbols $\sigma_{\alpha, \beta}$ for $\alpha, \beta \in \mathcal{K}_0(\mathbb{Q}(i))$ for the bimodule operation. The Lipschitz constant for the symbol $+$ is 2, while the Lipschitz constant of $\sigma_{\alpha, \beta}$ is $\|\alpha\| \|\beta\|$. An alternative description of operator spaces as metric structures—which fits in the framework of continuous logic [2, 10]—has been provided in [9, Section 3.3] and [13, Appendix B].

It is worth noting that the space X can be described as the set of $x \in \mathcal{K} \overline{\otimes} X$ such that $1.x = x$. Moreover a linear map $\phi : \mathcal{K} \overline{\otimes} X \rightarrow \mathcal{K} \overline{\otimes} Y$ that respects

the \mathcal{K}_0 -bimodule operations satisfies

$$\phi \left(\sum_i \gamma_i \otimes x_i \right) = \sum_i \gamma_i \otimes \phi(x)$$

and therefore is the amplification of a linear map from X to Y . Hence when operator spaces are regarded as \mathcal{L}_{OS} -structures, embeddings and isomorphisms as defined in Subsection 2.2 correspond, respectively, to completely isometric linear maps and completely isometric linear isomorphisms.

If X and Y are operator spaces, then the space $CB(X, Y)$ of completely bounded linear maps from X to Y is canonically endowed with an operator space structure obtained by identifying isometrically $M_n(CB(X, Y))$ with $CB(M_n(X), M_n(Y))$ with the completely bounded norm. Any linear functional ϕ on an operator system X is automatically completely bounded with $\|\phi\|_{cb} = \|\phi\|$. Therefore the dual space X^* of X can be regarded as the operator space $CB(X, \mathbb{C})$.

If X and Y are operator spaces, then their ∞ -sum $X \oplus^\infty Y$ is the operator system supported on the algebraic direct sum $X \oplus Y$ endowed with norms

$$\|(x, y)\|_{M_n(X \oplus^\infty Y)} = \max \left\{ \|x\|_{M_n(X)}, \|y\|_{M_n(Y)} \right\}.$$

The ∞ -sum of a sequence of operator spaces is defined analogously.

The 1-sum $X \oplus^1 Y$ is the operator system obtained by identifying $X \oplus Y$ with $(X^* \oplus^\infty Y^*)^*$. In formulas if $x, y \in M_n(X \oplus Y)$ then

$$\begin{aligned} \|(x, y)\|_{M_n(X \oplus^1 Y)} &= \|(x, y)\|_{CB(X^* \oplus Y^*, M_n)} \\ &= \sup \|(id_{M_n} \otimes \phi)(x) + (id_{M_n} \otimes \psi)(y)\|_{M_n \otimes M_k} \end{aligned}$$

where ϕ, ψ range over the unit balls of $CB(X^*, M_k)$ and $CB(Y^*, M_k)$ and k ranges in \mathbb{N} . Equivalently the norm on $X \oplus^1 Y$ can be described as

$$\|(x, y)\| = \sup_{u, v} \|(id_{M_n} \otimes u)(x) + (id_{M_n} \otimes v)(y)\|$$

where u, v range over all completely contractive maps from X and Y into $B(H)$; see [29, Section 2.6]. In analogous fashion one can define the 1-sum and the ∞ -sum of a sequence of operator spaces.

We denote the sum

$$\overbrace{\mathbb{C} \oplus^1 \mathbb{C} \oplus^1 \dots \oplus^1 \mathbb{C}}^{n \text{ times}}$$

by $\ell^1(n)$ and the sum

$$\overbrace{\mathbb{C} \oplus^\infty \mathbb{C} \oplus^\infty \dots \oplus^\infty \mathbb{C}}^{n \text{ times}}$$

by $\ell^\infty(n)$. Moreover we denote by $\bar{e} = (e_i)$ the canonical basis of $\ell^1(n)$ and by \bar{e}^* its dual basis of $\ell^\infty(n)$.

2.5. M_n -spaces. In this subsection we recall the definition and basic properties of M_n -spaces as defined in [21, Chapter I]. A *matricial n -norm* on a space X is a norm on $M_n(X)$ such that

$$\|\alpha.x.\beta\| \leq \|\alpha\| \|x\| \|\beta\|$$

for $\alpha, \beta \in M_n$ and $x \in M_n(X)$. Such a norm induces a norm on $M_k(X)$ for $k \leq n$ via the inclusion

$$x \mapsto \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}.$$

An L^∞ -*matrix- n -norm* is a matricial n -norm satisfying moreover

$$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\| = \max \{ \|x\|, \|y\| \}$$

for $k \leq n$, $x \in M_k(X)$, and $y \in M_{n-k}(Y)$.

Observe that M_n has a natural n -norm obtained by identifying $M_n(M_n)$ with $M_n \otimes M_n$ (spatial tensor product). More generally if K is a compact Hausdorff space then $C(K, M_n)$ is n -normed by identifying $M_n(C(K, M_n))$ with $C(K, M_n \otimes M_n)$. In particular $\ell^\infty(\mathbb{N}, M_n)$ has a natural n -norm obtained by the identification with $C(\beta\mathbb{N}, M_n)$.

If X, Y are n -normed spaces, then a linear map $\phi : X \rightarrow Y$ is *n -bounded* if $id_{M_n} \otimes \phi : M_n(X) \rightarrow M_n(Y)$ is bounded, and $\|\phi\|_n$ is by definition $\|id_{M_n} \otimes \phi\|$. The notions of n -contraction and n -isometry are defined similarly. Define $nB(X, Y)$ to be the space of n -bounded linear functions from X to Y with norm $\|\cdot\|_n$. Identifying $M_n(X^*)$ with $nB(X, \mathbb{C})$ isometrically defines an L^∞ -matrix- n -norm on the dual X^* of X . The same argument allows one to define an L^∞ -matrix-norm on the second dual X^{**} .

An L^∞ -matricially- n -normed space is called an *M_n -space* if it satisfies any of the following equivalent definitions—see [21, Théorème I.1.9]:

- (1) There is an n -isometry from X to $B(H)$;
- (2) The canonical inclusion of X into X^{**} is isometric;
- (3) $\|\sum_i \alpha_i x_i \beta_i\| \leq \|\sum_i \alpha_i \alpha_i^*\|^{\frac{1}{2}} \max_i \|x_i\| \|\sum_i \beta_i^* \beta_i\|^{\frac{1}{2}}$ for any $x_i \in M_n(X)$, $\alpha_i, \beta_i \in M_n$;
- (4) there is an n -isometry from X to $C(K, M_n)$ for some compact Hausdorff space K .

Clearly the case $n = 1$ gives the usual notion of Banach space. Characterization (3) allows one to show that M_n -spaces can be seen as structures in a suitable language \mathcal{L}_{M_n} . This is the same as the language for operator space described in Subsection 2.4 where one replaces \mathcal{K}_0 with M_n . When M_n -spaces are regarded as \mathcal{L}_{M_n} -spaces, embeddings and isomorphisms as defined in Subsection 2.2 correspond, respectively, to n -isometric linear maps and n -isometric linear isomorphisms.

The notions of quotient and subspace of an M_n -space can be defined analogously as in the case of operator spaces. Similarly the constructions of 1-sum and ∞ -sum can be performed in this context. More details can be found in [21, Section I.2]. We will use the same notations for the 1-sum

and ∞ -sum of operator spaces and M_n -spaces. This will be clear from the context and no confusion should arise.

For later use we recall the following observation; see [21, Remarque I.1.5]. Suppose that X is a finite-dimensional M_n -space, and that \bar{b}, \bar{b}^* is a bi-orthonormal system for X . Then the n -norm on X admits the following expression:

$$\left\| \sum_i \alpha_i \otimes b_i \right\| = \sup \left\{ \left\| \sum_i \alpha_i \otimes \beta_i \right\| : \beta_i \in M_n, \left\| \sum_i \alpha_i \otimes b_i^* \right\| \leq 1 \right\}$$

for $\alpha_i \in M_n$. In particular if \bar{e} is the canonical basis of $\ell^1(n)$ with dual basis \bar{e}^* of $\ell^\infty(n)$, then we obtain

$$\left\| \sum_i \alpha_i \otimes e_i \right\| = \sup \left\{ \left\| \sum_i \alpha_i \otimes \beta_i \right\| : \beta_i \in M_n, \|\beta_i\| \leq 1 \right\}.$$

Similar expressions hold for the matrix norms in operator spaces; see [29].

In the following we will often use tacitly the fact that finite-dimensional M_n -spaces can be approximated by subspaces of finite ∞ -sums of copies of M_n .

Lemma 2.11. Suppose that X is a finite-dimensional M_n -space and $\varepsilon > 0$. Then there is $k \in \mathbb{N}$ and an injective linear n -contraction

$$\phi : X \rightarrow \overbrace{M_n \oplus^\infty \cdots \oplus^\infty M_n}^{k \text{ times}}$$

such that $\|\phi^{-1}\|_n \leq 1 + \varepsilon$.

In Lemma 2.11 the map ϕ is not assumed to be injective. The expression $\|\phi^{-1}\|_n$ denotes the n -norm of ϕ^{-1} when regarded as a map from the range of ϕ to X . Similar conventions will be adopted in the rest of the present paper. We conclude by recalling that the natural analog of the Hahn-Banach theorem holds for M_n -spaces. Such an analog asserts that M_n is an injective element in the category of M_n -spaces with n -contractive maps as morphisms; see [21, Proposition I.1.16].

3. THE FRAÏSSÉ CLASS OF FINITE-DIMENSIONAL M_n -SPACES

The purpose of this section is to show that the class \mathcal{M}_n of finite-dimensional M_n -spaces is a complete Fraïssé class as in Definition 2.5. This will allow us to consider the corresponding Fraïssé limit as in Theorem 2.9. The case $n = 1$ of these results recovers the already known fact that finite-dimensional Banach spaces form a complete Fraïssé class. This has been shown by Ben Yaacov [1, Section 3.3] building on previous works of Henson (unpublished) and Kubis-Solecki [19]. For Banach spaces the limit is the Gurarij Banach space, introduced by Gurarij in [14] and proved to be unique up to isometric isomorphism by Lusky in [23].

3.1. Amalgamation property. The properties (JEP) and (HP) as in Definition 2.2 are clear for \mathcal{M}_n . We now show that \mathcal{M}_n has (AP). The proof is analogous to the one for Banach spaces, and consists in showing that the category of finite-dimensional M_n -spaces has pushouts; see [12, Lemma 2.1].

Lemma 3.1. Suppose that X, Y are M_n -spaces, $\delta \geq 0$, and $f : X \rightarrow Y$ is a linear injective n -contraction such that $\|f^{-1}\|_n \leq 1 + \delta$. Then there is an M_n -space Z and n -isometric linear maps $i : X \rightarrow Z$ and $j : Y \rightarrow Z$ such that $\|j \circ f - i\|_n \leq \delta$.

Proof. Define δX to be the M_n -space structure on X given by the norm

$$\|[x_{ij}]\|_{M_n(\delta X)} = \delta \|[x_{ij}]\|_{M_n(X)}.$$

Let \widehat{Z} be the 1-sum $X \oplus^1 Y \oplus^1 \delta X$, and Z be the quotient of \widehat{Z} by the subspace

$$N = \left\{ (-z, f(z), z) \in \widehat{Z} : z \in X \right\}.$$

Finally let $i : X \rightarrow Z$ and $j : Y \rightarrow Z$ the embeddings given by

$$x \mapsto (x, 0, 0) + N$$

and

$$y \mapsto (0, y, 0) + N.$$

We claim that i and j satisfy the desired conclusions. In fact it is clear that i and j are n -contractions such that $\|i \circ f - j\|_n \leq 1 + \delta$. We will now show that i is an n -isometry. The proof that j is an n -isometry is similar. Suppose that $x \in M_n(X)$ consider a linear n -contraction $\phi : X \rightarrow M_n$ such that $\|\phi(x)\|_{M_n \otimes M_n} = \|x\|_{M_n(X)}$. Observe that

$$\frac{1}{1+\delta} (\phi \circ f^{-1}) : f[X] \rightarrow M_n$$

is an linear n -contraction and hence it extends to a linear n -contraction $\psi : f[X] \rightarrow M_n$. Similarly the map

$$\frac{\delta}{1+\delta} \phi : \delta X \rightarrow M_n$$

is a linear n -contraction. Therefore we have that for every $z \in M_n(X)$

$$\begin{aligned} & \|(x - z, f(z), z)\|_{M_n(X \oplus^1 Y \oplus^1 \delta X)} \\ & \geq \left\| \phi(x - z) + \psi(z) + \frac{\delta}{1+\delta} \phi(z) \right\|_{M_n \otimes M_n} \\ & = \left\| \phi(x) - \phi(z) + \frac{1}{1+\delta} \phi(z) + \frac{\delta}{1+\delta} \phi(z) \right\|_{M_n \otimes M_n} \\ & = \|\phi(x)\|_{M_n \otimes M_n} = \|x\|_{M_n(X)}. \end{aligned}$$

This concludes the proof that i is an n -isometry. \square

In particular Lemma 3.1 for $\delta = 0$ shows that the class \mathcal{M}_n has (AP).

3.2. The Fraïssé metric space. We fix now $k \in \mathbb{N}$ and consider the space $\mathcal{M}_n(k)$ of pairs (\bar{a}, X) such that X is a k -dimensional M_n -space and \bar{a} is a linear basis of X . Two such pairs (\bar{a}, X) and (\bar{b}, Y) are identified if there is an n -isometry ϕ from X to Y such that $\phi(\bar{a}) = \bar{b}$. For simplicity we will write an element (\bar{a}, X) of $\mathcal{M}_n(k)$ simply by \bar{a} , and denote X by $\langle \bar{a} \rangle$. Our goal is to compute the Fraïssé metric in $\mathcal{M}_n(k)$ as in Definition 2.4. The following result gives an equivalent characterization of such a metric. The case $n = 1$ is a result of Henson (unpublished) that can be found in [1, Fact 3.2]. We denote by $\ell^1(k)$ the k -fold 1-sum of \mathbb{C} by itself in the category of M_n -spaces with canonical basis \bar{e} . An explicit formula for the corresponding norm has been recalled at the end of Section 2.5.

Proposition 3.2. Suppose that $\bar{a}, \bar{b} \in \mathcal{M}_n(k)$ and $M > 0$. The following statements are equivalent:

- (1) $d_{\mathcal{M}_n}(\bar{a}, \bar{b}) \leq M$;
- (2) For every n -contractive $u : \langle \bar{a} \rangle \rightarrow M_n$ there is an n -contractive $v : \langle \bar{b} \rangle \rightarrow M_n$ such that the linear function $w : \ell^1(k) \rightarrow M_n$ defined by $w(e_i) = u(a_i) - v(b_i)$ has n -norm at most M , and vice versa.

Proof. After normalizing we can assume that $M = 1$. We will denote $\langle \bar{a} \rangle$ by X and $\langle \bar{b} \rangle$ by Y .

(1) \Rightarrow (2): Suppose that $d_{\mathcal{M}_n}(\bar{a}, \bar{b}) \leq 1$. Then there are n -isometries $\phi : \langle \bar{a} \rangle \rightarrow Z$ and $\psi : \langle \bar{b} \rangle \rightarrow Z$ for some M_n -space Z such that $\|\phi(a_i) - \psi(b_i)\| \leq 1$ for every $i \leq k$. Suppose that $u : X \rightarrow M_n$ is n -contractive. Consider the n -contractive map $u \circ \phi^{-1} : \phi[X] \rightarrow M_n$. By injectivity of M_n there is an n -contractive map $\eta : Z \rightarrow M_n$ extending $u \circ \phi^{-1}$. Define $v = \eta \circ \psi : Y \rightarrow M_n$ and observe that it is n -contractive. Define now $w : \ell^1(k) \rightarrow M_n$ by $w(e_i) = u(a_i) - v(b_i)$. We claim that w is n -contractive. In fact

$$\|\eta(\phi(e_i) - \psi(e_i))\| \leq 1$$

for every $i \leq k$. Therefore if $\alpha_i \in M_n$

$$\begin{aligned} \left\| (id_{M_n} \otimes w) \left(\sum_i \alpha_i \otimes e_i \right) \right\| &= \left\| \sum_i \alpha_i \otimes \eta(\phi(e_i) - \psi(e_i)) \right\| \\ &\leq \left\| \sum_i \alpha_i \otimes e_i \right\|_{M_n(\ell^1(k))}. \end{aligned}$$

The vice versa is proved analogously.

(2) \Rightarrow (1): Conversely suppose that for every n -contractive $u : X \rightarrow M_n$ there is an n -contractive $v : Y \rightarrow M_n$ such that the linear function $w : \ell^1(k) \rightarrow M_n$ defined by $w(e_i) = u(a_i) - v(b_i)$ is n -contractive, and vice versa. Define \hat{Z} to be

$$X \oplus^1 Y \oplus^1 \ell^1(k).$$

Denote by N the closed subspace

$$\left\{ \left(-\sum_i \lambda_i a_i, \sum_i \lambda_i b_i, \sum_i \lambda_i e_i \right) : \lambda_i \in \mathbb{C} \right\}$$

of \widehat{Z} . Define Z to be the quotient of \widehat{Z} by N . Let ϕ be the composition of the canonical inclusion of X into \widehat{Z} with the quotient map from \widehat{Z} to Z . Similarly define $\psi : Y \rightarrow Z$. By the properties of 1-sums and quotients, ϕ and ψ are n -contractions. We claim that they are in fact n -isometries. We will only show that ϕ is an n -isometry, since the proof for ψ is entirely analogous. Suppose that $x \in M_n(X)$. Pick an n -contraction $u : X \rightarrow M_n$ such that

$$\|x\| = \|(id_{M_n} \otimes u)(x)\|.$$

By hypothesis there are n -contractions $v : Y \rightarrow M_n$ and $w : \ell^1(k) \rightarrow M_n$ such that $w(e_i) = u(e_i) - v(e_i)$. Therefore if $\alpha_i \in M_n$ then the norm of

$$\left(x - \sum_i \alpha_i \otimes a_i, \sum_i \alpha_i \otimes b_i, \sum_i \alpha_i \otimes e_i \right)$$

in $M_n(\widehat{Z})$ is bounded from below by the norm of

$$\begin{aligned} & (id_{M_n} \otimes u) \left(x - \sum_i \alpha_i \otimes a_i \right) \\ & + (id_{M_n} \otimes v) \left(\sum_i \alpha_i \otimes b_i \right) \\ & + (id_{M_n} \otimes w) \left(\sum_i \alpha_i \otimes e_i \right) \\ & = (id_{M_n} \otimes u)(x) \end{aligned}$$

which equals $\|x\|$. Since this is true for every $\alpha_i \in M_n$, ϕ is an n -isometry. Similarly ψ is an n -isometry. The proof is concluded by observing that $\|\phi(a_i) - \psi(b_i)\| \leq 1$ for every $i \leq k$. \square

Corollary 3.3. If $\bar{a}, \bar{b} \in \mathcal{M}_n(k)$ and $d_{\mathcal{M}_n}(\bar{a}, \bar{b}) \leq M$ then for every $\alpha_i \in M$

$$\left\| \sum_i \alpha_i \otimes a_i \right\| - \left\| \sum_i \alpha_i \otimes b_i \right\| \leq M \left\| \sum_i \alpha_i \otimes e_i \right\|$$

where \bar{e} is the canonical basis of $\ell^1(k)$.

An *Auerbach system* in a Banach space is a basis \bar{a} with dual basis \bar{a}^* such that $\|a_i\| = \|a_i^*\| = 1$. By analogy we say that an element \bar{a} of $\mathcal{M}_n(k)$ is *N-Auerbach* if $\|a_i\| \leq N$ and $\|a_i^*\| \leq N$ for every $i \leq k$. Denote by $\mathcal{M}_n(k, N)$ the set of N -Auerbach $\bar{a} \in \mathcal{M}_n(k)$. It follows from Corollary 3.3

that the set $\mathcal{M}_n(k, N)$ is closed in $\mathcal{M}_n(k)$. It can be easily verified that if $\bar{a} \in \mathcal{M}_n(k, N)$ and $\alpha_i \in M_n$ then

$$\left\| \sum_i \alpha_i \otimes a_i \right\| \leq N \left\| \sum_i \alpha_i \otimes e_i \right\|$$

and

$$\left\| \sum_i \alpha_i \otimes e_i \right\| \leq kN \left\| \sum_i \alpha_i \otimes \bar{e}_i \right\|$$

where \bar{e} is the canonical basis of $\ell^1(k)$.

If $\bar{a}, \bar{b} \in \mathcal{M}_n(k)$, denote by $\iota_{\bar{a}, \bar{b}}$ the linear isomorphism from $\langle \bar{a} \rangle$ to $\langle \bar{b} \rangle$ such that $\iota_{\bar{a}, \bar{b}}(a_i) = b_i$ for $i \leq k$. Define the n -bounded distance $d_{nb}(\bar{a}, \bar{b})$ to be $\|\iota_{\bar{a}, \bar{b}}\|_n \|\iota_{\bar{a}, \bar{b}}^{-1}\|_n$. (Observe that this is not an actual metric, but $\log(d_{nb})$ is a metric.)

In the following lemma we establish a precise relation between the n -bounded distance d_{nb} and the Fraïssé metric d_{nb} on $\mathcal{M}_n(k, N)$.

Proposition 3.4. Suppose that $\bar{a}, \bar{b} \in \mathcal{M}_n(k, N)$. Then

$$d_{nb}(\bar{a}, \bar{b}) \leq (1 + kNd_{\mathcal{M}_n}(\bar{a}, \bar{b}))^2$$

and

$$d_{\mathcal{M}_n}(\bar{a}, \bar{b}) \leq d_{nb}(\bar{a}, \bar{b}) - 1.$$

Proof. Suppose that $d_{\mathcal{M}_n}(\bar{a}, \bar{b}) \leq M$. Then by Corollary 3.3

$$\left\| \sum_i \alpha_i \otimes a_i \right\| - \left\| \sum_i \alpha_i \otimes b_i \right\| \leq M \left\| \sum_i \alpha_i \otimes e_i \right\|$$

for every $\alpha_i \in M_n$, where \bar{e} is the canonical basis of $\ell^1(k)$. Since \bar{a}, \bar{b} are N -Auerbach we have

$$\left\| \sum_i \alpha_i \otimes a_i \right\| \leq \left\| \sum_i \alpha_i \otimes b_i \right\| + M \left\| \sum_i \alpha_i \otimes e_i \right\| \leq (1 + kNM) \left\| \sum_i \alpha_i \otimes b_i \right\|$$

and similarly

$$\left\| \sum_i \alpha_i \otimes b_i \right\| \leq (1 + kNM) \left\| \sum_i \alpha_i \otimes \bar{e}_i \right\|.$$

Therefore

$$d_{nb}(\bar{a}, \bar{b}) \leq (1 + kNM)^2.$$

The other inequality is an immediate consequence of Lemma 3.1. \square

We can finally show that the space $(\mathcal{M}_n(k), d_{\mathcal{M}_n})$ is separable and complete. In view of Proposition 3.4 this can be proved by a standard argument; see for example [29, Theorem 21.1 and Remark 21.2]. A proof is included for the sake of completeness.

Proposition 3.5. The space $(\mathcal{M}_n(k), d_{\mathcal{M}_n})$ is separable and complete.

Proof. Observe that, by Proposition 3.4, if $(\bar{a}^{(m)})_{m \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{M}_n(k)$, then there is $N \in \mathbb{N}$ such that $\bar{a}^{(m)} \in \mathcal{M}_n(k, N)$ for every $n \in \mathbb{N}$. Therefore it is enough to show that, for every $N \in \mathbb{N}$, the space $(\mathcal{M}_n(k, N), d_{\mathcal{M}_n})$ is compact. Suppose that $(\bar{a}^{(m)})_{m \in \mathbb{N}}$ is a sequence in $\mathcal{M}_n(k, N)$. If $\alpha_i \in M_n$ then

$$\left(\left\| \sum_i \alpha_i \otimes a_i^{(m)} \right\| \right)_{m \in \mathbb{N}}$$

is a bounded sequence of complex numbers. Therefore after passing to a subsequence we can assume that such a sequence converges for any choice of $\alpha_i \in M_n(\mathbb{Q}(i))$. This is easily seen to imply that in fact such a sequence converges for any choice of $\alpha_i \in M_n$. Moreover the functions

$$(\alpha_1, \dots, \alpha_k) \mapsto \left\| \sum_i \alpha_i \otimes a_i^{(m)} \right\|$$

are equiuniformly continuous on the unit ball of M_n . Therefore, by the Ascoli-Arzelà theorem, after passing to a further subsequence we can assume that the convergence is uniform on the unit ball of M_n . We can now define an element \bar{a} of $\mathcal{M}_n(k)$ by setting

$$\left\| \sum_i \alpha_i \otimes a_i \right\| = \lim_{m \rightarrow +\infty} \left\| \sum_i \alpha_i \otimes a_i^{(m)} \right\|.$$

The abstract characterization of M_n -spaces shows that \bar{a} is indeed an element of $\mathcal{M}_n(k)$. By uniform convergence in the unit ball the sequence $(\bar{a}^{(m)})_{m \in \mathbb{N}}$ is such that $d_{nb}(\bar{a}^{(m)}, \bar{a}) \rightarrow 1$. Therefore $d_{\mathcal{M}_n}(\bar{a}^{(m)}, \bar{a}) \rightarrow 0$ by Proposition 3.4. \square

This concludes the proof that \mathcal{M}_n is a complete Fraïssé class.

3.3. The Fraïssé limit. We have verified that the class \mathcal{M}_n is a Fraïssé class in the sense of Definition 2.5. Therefore by Theorem 2.9 we can consider its Fraïssé limit. Observe that the \mathcal{M}_n -structures are precisely the M_n -spaces. We first provide a characterization for the Fraïssé limit of \mathcal{M}_n similar in spirit to the universal property defining the Gurarij Banach space.

Proposition 3.6. Suppose that Z is a separable M_n -space. The following statements are equivalent:

- (1) Z is the Fraïssé limit of the class \mathcal{M}_n ;
- (2) If $X \subset Y$ are finite-dimensional M_n -spaces, $\psi : X \rightarrow Z$ is a linear n -isometry, and $\varepsilon > 0$, then there is a linear function $\phi : Y \rightarrow Z$ extending ψ such that $\|\phi\|_n \|\phi^{-1}\|_n < 1 + \varepsilon$.

Proof. The proof is entirely analogous to the proof of [1, Theorem 3.3], and is presented here for convenience of the reader.

- (1) \Rightarrow (2): Suppose that Z is the Fraïssé limit of the class \mathcal{M}_n . Suppose that $X \subset Y$ are finite-dimensional M_n -spaces, $\phi : X \rightarrow Z$ is a linear n -isometry, and $\varepsilon > 0$. Fix $\delta > 0$ small enough. Consider also a basis (a_1, \dots, a_k) of X and a basis (b_1, \dots, b_m) of Y such that $b_i = a_i$ for $i \leq k$. Since Z is by assumption the Fraïssé limit of the class \mathcal{M}_n , there is a linear n -isometry $\hat{\phi} : Y \rightarrow Z$ such that $\|\phi(a_i) - \hat{\phi}(a_i)\| \leq \delta$ for every $i \leq k$. Define now $\psi : Y \rightarrow Z$ by setting $\psi(b_i) = \phi(a_i)$ for $i \leq k$ and $\psi(b_i) = \hat{\phi}(a_i)$ for $k < i \leq m$. A routine calculation shows that, for δ small enough, ψ satisfies the desired inequality.
- (2) \Rightarrow (1): Suppose now that Z satisfies condition (2). Consider $X \in \mathcal{M}_n(k)$, a finite l -tuple \bar{b} in X , $\psi \in \text{Stx}_{\mathcal{M}_n}(X, Z)$, and $\varepsilon > 0$. By [1, Lemma 2.16] in order to show that Z is the Fraïssé limit of \mathcal{M}_n it is enough to find $\varphi \in \text{Stx}_{\mathcal{M}_n}^{<\psi}(X, Z)$ with the following property: for every $i \leq l$ there is $y \in Z$ such that $\varphi(b_i, y) < \varepsilon$. By [1, Lemma 2.8(iii)] after enlarging X , and decreasing ε we can assume that there is a finite m -tuple \bar{c} in X and an n -isometric linear map $f : \langle \bar{c} \rangle \rightarrow Z$ such that $\psi \geq f|_{\bar{c}} + \varepsilon$. (Recall our convention of identifying partial isomorphisms between \mathcal{L} -structures with the corresponding approximate isomorphisms.) Denote by $\bar{b}\bar{c}$ the concatenation of the tuples \bar{b} and \bar{c} . By assumption if $\delta > 0$, then we can extend f to a linear map $f : X \rightarrow Z$ satisfying $\|f\|_n \|f^{-1}\|_n < 1 + \delta$. In view of Proposition 3.4 by choosing δ small enough one can ensure that

$$d_{\mathcal{M}_n}(\bar{b}\bar{c}, f(\bar{b}\bar{c})) < \varepsilon.$$

Therefore by the definition of the Fraïssé metric $d_{\mathcal{M}_n}$ (Definition 2.4) there is

$$\varphi \in \text{Stx}_{\mathcal{M}_n}(\langle \bar{b}\bar{c} \rangle, \langle f(\bar{b}\bar{c}) \rangle) \subset \text{Stx}_{\mathcal{M}_n}(X, Z)$$

such that $\varphi(b_i, f(b_i)) < \varepsilon$ for $i \leq l$ and $\varphi(c_j, f(c_j)) < \varepsilon$ for $j \leq m$. Observe that such a φ satisfies

$$\psi \geq f|_{\bar{c}} + \varepsilon > \varphi|_{\bar{c} \times f(\bar{c})} \geq \varphi.$$

This concludes the proof. \square

In view of Proposition 3.6 the following theorem is an immediate consequence of 2.9 and the fact that \mathcal{M}_n is a complete Fraïssé class.

Theorem 3.7. There is a separable M_n -space \mathbb{G}_n with the following property: If $X \subset Y$ are finite-dimensional M_n -spaces, $\psi : X \rightarrow \mathbb{G}_n$ is a linear n -isometry, and $\varepsilon > 0$, then there is a linear function $\phi : Y \rightarrow \mathbb{G}_n$ extending ψ such that $\|\phi\|_n \|\phi^{-1}\|_n < 1 + \varepsilon$. Any two separable M_n -spaces with such a property are n -isometrically isomorphic. Moreover \mathbb{G}_n contains any separable M_n -space as a subspace, and has the following homogeneity property: If $X \subset \mathbb{G}_n$ is finite-dimensional, $\phi : X \rightarrow \mathbb{G}_n$ is a linear n -isometry, and $\varepsilon > 0$, then there is a surjective n -isometry $\psi : \mathbb{G}_n \rightarrow \mathbb{G}_n$ such that $\|\psi|_X - \phi\|_n < \varepsilon$.

Clearly for $n = 1$ one obtains a Banach space which is isometrically isomorphic to the Gurarij space. In Subsection 4.5 we will show that for $n \leq k$ the spaces \mathbb{G}_n and \mathbb{G}_k are n -isometrically isomorphic.

4. THE NONCOMMUTATIVE GURARIJ SPACE

4.1. MIN and MAX spaces. Clearly any operator space can be canonically regarded as an M_n -space. Conversely if X is an M_n -space, then there are two canonical ways to regard X as an operator space. It is natural to call an operator space structure \widehat{X} on X *compatible* if the map $X \mapsto \widehat{X}$ is an n -isometry. The *minimal and maximal* compatible operator space structures $\text{MIN}_n(X)$ and $\text{MAX}_n(X)$ on an M_n -space are defined by the formulas

$$\|x\|_{M_k(\text{MIN}_n(X))} = \sup_{\phi} \|(id_{M_k} \otimes \phi)(x)\|_{M_k \otimes M_n}$$

where ϕ varies among all n -contractions from X to M_n , and

$$\|x\|_{M_k(\text{MAX}_n(X))} = \sup_u \|(id_{M_k} \otimes u)(x)\|_{M_k(B(H))}$$

where u varies among all n -contractions from X to $B(H)$. These are introduced in [21, Section I.3] as a generalization of the minimal and maximal quantization of a Banach space as in [8, Section 3.3]; see also [26, Section 2]. If X is an operator space then we define $\text{MIN}_n(X)$ and $\text{MAX}_n(X)$ to be the structures defined above starting from X regarded just as M_n -space. This is consistent with the terminology used in [25, 26].

The names MIN and MAX are suggestive of the following property; see [21, Proposition I.3.1]. If \widehat{X} is a compatible operator space structure on X then the identity maps

$$\text{MAX}_n(X) \rightarrow \widehat{X} \rightarrow \text{MIN}_n(X)$$

are completely bounded. The operator space structures MIN and MAX are characterized by the following universal property; see [21, Proposition I.3.6 and Proposition I.3.7]. If Z is an operator space and $u : Z \rightarrow X$ is a linear map, then $u : Z \rightarrow X$ is n -bounded if and only if $u : Z \rightarrow \text{MIN}_n(X)$ is completely bounded, and in such case

$$\|u : Z \rightarrow \text{MIN}_n(X)\|_{cb} = \|u : Z \rightarrow X\|_n.$$

Similarly if Z is an operator space and $u : X \rightarrow Z$ is a linear map, then $u : X \rightarrow Z$ is n -bounded if and only if $u : \text{MAX}_n(X) \rightarrow Z$ is completely bounded, and in such case

$$\|u : \text{MAX}_n(X) \rightarrow Z\|_{cb} = \|u : X \rightarrow Z\|_n.$$

Remark 4.1. In the following we will always consider an M_n -space X as an operator system endowed with its minimal compatible operator system structure.

It is worth noting at this point that all the proofs of Section 3 go through without change when M_n -spaces are regarded as operator spaces with their minimal compatible operator space structure. This easily follows from the properties of the minimal quantization recalled above.

4.2. Exact and 1-exact operator spaces. Suppose that E and F are two finite-dimensional operator spaces. Define $d_{cb}(E, F)$ to be the infimum of $\|\phi\|_{cb} \|\phi^{-1}\|_{cb}$ when ϕ ranges over all linear isomorphisms from E to F . The *exactness constant* $ex(E)$ of a finite-dimensional operator space is the infimum of $d_{cb}(E, F)$ where F ranges among all subspaces of M_n for $n \in \mathbb{N}$. Equivalently one can define $ex(E)$ to be the limit for $n \rightarrow +\infty$ of the decreasing sequence

$$\|id_E : \text{MIN}_n(E) \rightarrow E\|_{cb},$$

where id_E denotes the identity map of E . If X is a not necessarily finite-dimensional operator space, then its exactness constant $ex(X)$ is the supremum of $ex(E)$ where E ranges over all finite-dimensional subspaces of E .

An operator space is *exact* if it has finite exactness constant, and *1-exact* if it has exactness constant 1. For C^* -algebras exactness is equivalent to 1-exactness, which is in turn equivalent to several other properties; see [3, Section IV.3.4]. Exactness is a fundamental notion in the theory of C^* -algebras and operator spaces. It is a purely noncommutative phenomenon: there is no Banach space analog of nonexactness. In fact every Banach space—and in fact every M_n -space—is 1-exact. More information and several equivalent characterizations of exactness can be found in [28] and [29, Chapter 17].

In the following we will denote by \mathcal{E}_1 the class of finite-dimensional 1-exact operator spaces. Moreover we will denote by $\mathcal{M}_\infty^0 \subset \mathcal{E}_1$ the class of operator spaces that admit a completely isometric embedding into M_n for some $n \in \mathbb{N}$. Our goal is to show that \mathcal{E}_1 is a Fraïssé class.

4.3. Amalgamation of 1-exact operator spaces. It is clear that \mathcal{E}_1 has (HP) from Definition 2.2. It remains to verify that \mathcal{E}_1 satisfies (AP). This will give (JEP) as consequence, since the trivial operator space $\{0\}$ embeds in every element of \mathcal{E}_1 .

We recall that if (Z_n) is a direct sequence of operator spaces with completely isometric linear maps $\phi_n : Z_n \rightarrow Z_{n+1}$ one can define the *direct limit* $\lim_{(\phi_n)} Z_n$ with canonical completely isometric linear maps $\sigma_k : Z_k \rightarrow \lim_{(\phi_n)} Z_n$ in the following way. Let $W = \ell^\infty(\mathbb{N}, (Z_n))$ be the space of sequences $(z_n) \in \prod_n Z_n$ with $\sup_n \|z_n\| < +\infty$. Define an operator seminorm structure on \widehat{W} in the sense of [4, 1.2.16] by setting

$$\rho_k((z_n)_{n \in \mathbb{N}}) = \limsup_{n \rightarrow +\infty} \|z_n\|_{M_k(Z_n)}$$

for $k \in \mathbb{N}$ and $z_n \in M_k(Z_n)$. Finally define W to be the operator space associated with such an operator seminorm structure on \widehat{W} . For n, m let

$\phi_{n,n} = id_{Z_n}$, $\phi_{n,m} = \phi_{m-1} \circ \cdots \circ \phi_n$ if $n < m$, and $\phi_{n,m} = 0$ otherwise. Define the maps $\sigma_k : Z_k \rightarrow W$ by

$$\begin{aligned} Z_k &\rightarrow W \\ x &\mapsto (\phi_{k,n}(x))_{n \in \mathbb{N}}. \end{aligned}$$

Finally set $\lim_{(\phi_n)} Z_n$ to be the closure inside W of the union of the images of Z_k under σ_k for $k \in \mathbb{N}$. It is clear that if for every $k \in \mathbb{N}$ the space Z_k is 1-exact, then $\lim_{(\phi_n)} Z_n$ is 1-exact.

The proof of the following proposition is inspired by [6, Theorem 4.7] and [25, Theorem 1.1].

Proposition 4.2. Suppose that $X_0 \subset X$ and Y are finite-dimensional 1-exact operator spaces, $\delta \geq 0$, $\varepsilon > 0$, and $f : X_0 \rightarrow Y$ is a complete contraction such that $\|f^{-1}\|_{cb} \leq 1 + \delta$. Then there exists a 1-exact separable operator space Z and linear complete isometries $j : Y \rightarrow Z$ and $i : X \rightarrow Z$ such that $\|j \circ f - i|_{X_0}\|_{cb} \leq \delta + \varepsilon$.

Proof. Without loss of generality we can assume that $\varepsilon \leq 1$. We will construct by recursion on k sequences $(n_k)_{k \in \mathbb{N}}$, $(Z_k)_{k \in \mathbb{N}}$, $i_k : X \rightarrow Z_k$, $j_k : Y \rightarrow Z_k$, $\phi_k : Z_k \rightarrow Z_{k+1}$ such that

- (1) $(n_k)_{k \in \mathbb{N}}$ is nondecreasing,
- (2) Z_k is an M_{n_k} -space,
- (3) i_k and j_k are injective completely contractive linear maps,
- (4) ϕ_k is a completely isometric linear map,
- (5) $\|i_k^{-1}\|_{cb} \leq 1 + \varepsilon 2^{-k}$, $\|j_k^{-1}\|_{cb} \leq 1 + \varepsilon 2^{-k}$,
- (6) $\|\phi_k \circ i_k - i_{k+1}\|_{cb} \leq 1 + \varepsilon 2^{-k}$, $\|\phi_k \circ j_k - j_{k+1}\|_{cb} \leq 1 + \varepsilon 2^{-k}$, and
- (7) $\|j_k \circ f - (i_k)|_{X_0}\|_{cb} \leq \delta + (2\varepsilon) \sum_{i < k} 2^{-i}$.

We can apply Lemma 3.1 and Lemma 2.11 to define n_1, Z_1, i_1 , and j_1 . Suppose that n_k, Z_k, i_k, j_k , and ϕ_{k-1} have been defined for $k \leq m$. By Lemma 2.11 we can pick $n_{m+1} \geq n_m$ and injective completely contractive maps $\theta_X : X \rightarrow M_{n_{m+1}}$ and $\theta_Y : Y \rightarrow M_{n_{m+1}}$ such that $\|\theta_X^{-1}\|_{cb} \leq 1 + \varepsilon 2^{-2(m+1)}$ and $\|\theta_Y^{-1}\|_{cb} \leq 1 + \varepsilon 2^{-(m+1)}$. By injectivity of $M_{n_{m+1}}$ there are complete contractions $\alpha_X, \alpha_Y : Z_m \rightarrow M_{n_{m+1}}$ such that

$$\alpha_X \circ i_m = \frac{1}{1 + \varepsilon 2^{-m}} \theta_X \quad \text{and} \quad \alpha_Y \circ j_m = \frac{1}{1 + \varepsilon 2^{-m}} \theta_Y.$$

Define W to be $\text{MIN}_{n_{m+1}}(Z_m \oplus^\infty M_{n_{m+1}})$. Define linear maps

$$\begin{aligned} \widehat{\theta}_X : X &\rightarrow W \\ x &\mapsto (i_m(x), \theta_X(x)), \end{aligned}$$

$$\begin{aligned} \widehat{\theta}_Y : Y &\rightarrow W \\ y &\mapsto (j_m(y), \theta_Y(y)), \end{aligned}$$

and

$$\begin{aligned}\widehat{\alpha}_X : Z_m &\rightarrow W \\ z &\mapsto (z, \alpha_X(z)), \\ \widehat{\alpha}_Y : Z_m &\rightarrow W \\ z &\mapsto (z, \alpha_Y(z)).\end{aligned}$$

Observe that $\widehat{\alpha}_X, \widehat{\alpha}_Y$ are completely isometric, while $\widehat{\theta}_X$ and $\widehat{\theta}_Y$ are completely contractive with

$$\left\| \widehat{\theta}_X^{-1} \right\|_{cb} \leq \left\| \theta_X^{-1} \right\|_{cb} \leq 1 + \varepsilon 2^{-(m+1)}$$

and

$$\left\| \widehat{\theta}_Y^{-1} \right\|_{cb} \leq \left\| \theta_Y^{-1} \right\|_{cb} \leq 1 + \varepsilon 2^{-(m+1)}.$$

Note also that

$$\left\| \widehat{\theta}_X - \widehat{\alpha}_X \circ i_m \right\|_{cb} \leq \varepsilon 2^{-m} \quad \text{and} \quad \left\| \widehat{\theta}_Y - \widehat{\alpha}_Y \circ i_m \right\|_{cb} \leq \varepsilon 2^{-m}.$$

Define now

$$N = \{(-(z_0 + z_1), \widehat{\alpha}_X(z_0), \widehat{\alpha}_Y(z_1)) \in Z_m \oplus W \oplus W : z_0, z_1 \in Z_m\}.$$

Let Z_{m+1} be

$$\text{MIN}_{n_{m+1}}((Z_m \oplus^1 W \oplus^1 W)/N).$$

Consider the first coordinate inclusion $\phi_m : Z_m \rightarrow Z_{m+1}$ of Z_m into Z_{m+1} . Similarly define $\psi_X, \psi_Y : W \rightarrow Z_{m+1}$ to be the second and third coordinate inclusions. Arguing as in the proof of Lemma 3.1 one can verify directly that ϕ_m, ψ_X, ψ_Y are complete isometries. Alternatively one can use [25, Lemma 2.4] together with the properties of MIN. Observe that $\widehat{\alpha}_X \circ \phi_m = \psi_X$ and $\widehat{\alpha}_Y \circ \phi_m = \psi_Y$. Define now linear complete contractions

$$i_{m+1} := \psi_X \circ \widehat{\theta}_X : X \rightarrow Z_{m+1} \quad \text{and} \quad j_{m+1} := \psi_Y \circ \widehat{\theta}_Y : Y \rightarrow Z_{m+1}.$$

Observe that

$$\left\| i_{m+1}^{-1} \right\|_{cb} \leq \left\| \widehat{\theta}_X^{-1} \right\|_{cb} < 1 + \varepsilon 2^{-(m+1)}$$

and

$$\left\| j_{m+1}^{-1} \right\|_{cb} \leq \left\| \widehat{\theta}_Y^{-1} \right\|_{cb} < 1 + \varepsilon 2^{-(m+1)}.$$

Moreover

$$\begin{aligned}\left\| \phi_m \circ i_m - i_{m+1} \right\|_{cb} &= \left\| \phi_m \circ i_m - \psi_X \circ \widehat{\theta}_X \right\|_{cb} \\ &\leq \left\| \phi_m \circ i_m - \psi_X \circ \widehat{\alpha}_X \circ i_m \right\|_{cb} + \varepsilon 2^{-m} \\ &= \varepsilon 2^{-m}.\end{aligned}$$

Similarly

$$\left\| \psi_m \circ j_m - j_{m+1} \right\|_{cb} \leq \varepsilon 2^{-m}.$$

Finally we have

$$\begin{aligned}
\|i_{m+1} - j_{m+1} \circ f\|_{cb} &= \|\psi_X \circ \widehat{\theta}_X - \psi_Y \circ \widehat{\theta}_Y \circ f\|_{cb} \\
&\leq \|\psi_X \circ \widehat{\alpha}_X \circ i_m - \psi_Y \circ \widehat{\alpha}_Y \circ j_m \circ f\|_{cb} + (2\varepsilon) 2^{-m} \\
&\leq \|\phi_m \circ i_m - \phi_m \circ j_m \circ f\| + (2\varepsilon) 2^{-m} \\
&\leq \|i_m - j_m \circ f\| + (2\varepsilon) 2^{-m} \\
&\leq \delta + (2\varepsilon) \sum_{i \leq m} 2^{-i}.
\end{aligned}$$

This concludes the recursive construction. Let now Z be $\lim_{(\phi_k)} Z_k$ with canonical linear complete isometries $\sigma_k : Z_k \rightarrow Z$. Consider also the embeddings $i : X \rightarrow Z$ and $j : Y \rightarrow Z$ defined by

$$i := \lim_{k \rightarrow +\infty} \sigma_k \circ i_k \quad \text{and} \quad j := \lim_{k \rightarrow +\infty} \sigma_k \circ j_k.$$

It is easily seen as in the proof of [6, Theorem 4.7] that Z is a 1-exact separable operator space, and i, j are well defined completely isometric linear maps such that $\|j \circ f - i|_{X_0}\|_{cb} \leq \delta + 2\varepsilon$. \square

In particular Proposition 4.2 for $\delta = 0$ shows that the class \mathcal{E}_1 has (NAP). It is not difficult to modify the proof above to show that the conclusions of Proposition 4.2 hold even when $X_0 \subset X$ and Y are not necessarily finite-dimensional separable 1-exact operator spaces. Moreover one can obtain Z to be an $\mathcal{OL}_{\infty,1+}$ space in the sense of [7, 15].

4.4. The Fraïssé metric space. Fix $k \in \mathbb{N}$ and denote by $\mathcal{E}_1(k)$ the space of pairs (\bar{a}, X) such that X is a k -dimensional 1-exact operator space and \bar{a} is a basis of X . Two such pairs (\bar{a}, X) and (\bar{b}, Y) are identified if there is a complete isometry ϕ from X to Y such that $\phi(\bar{a}) = \bar{b}$. To simplify the notation the pair (\bar{a}, X) will be simply denoted \bar{a} , where we set $X = \langle \bar{a} \rangle$. Denote by $d_{\mathcal{E}_1}$ the Fraïssé metric on $\mathcal{E}_1(k)$ as in Definition 2.4. We further denote by $\mathcal{M}_{\infty}^0(k)$ the subspace of $\mathcal{E}_1(k)$ consisting of pairs (\bar{a}, X) such that X admits a completely isometric embedding into M_n for some $n \in \mathbb{N}$. Let $\ell^1(k)$ be the k -fold 1-sum of \mathbb{C} by itself in the category of operator spaces. A similar proof as the one of Proposition 3.2 gives the following:

Proposition 4.3. Suppose that $\bar{a}, \bar{b} \in \mathcal{E}_1(k)$ and $M > 0$. If $d_{\mathcal{E}_1}(\bar{a}, \bar{b}) \leq M$ then for every $n \in \mathbb{N}$ and every completely contractive $u : X \rightarrow M_n$ there is a completely contractive $v : Y \rightarrow M_n$ such that the linear function $w : \ell^1(k) \rightarrow B(H)$ defined by $w(e_i) = u(a_i) - v(b_i)$ has completely bounded norm at most M , and vice versa.

Corollary 4.4. Suppose that $\bar{a}, \bar{b} \in \mathcal{E}_1(k)$ and $M > 0$. If $d_{\mathcal{E}_1}(\bar{a}, \bar{b}) \leq M$ then for every $n \in \mathbb{N}$ and $\alpha_i \in M_n$

$$\left\| \sum_i \alpha_i \otimes a_i \right\| - \left\| \sum_i \alpha_i \otimes b_i \right\| \leq M \left\| \sum_i \alpha_i \otimes e_i \right\|$$

where \bar{e} denotes the canonical basis of $\ell^1(k)$.

As in Subsection 3.2 we define an element \bar{a} of \mathcal{E}_1 to be *N-Auerbach* if $\|a_i\| \leq N$ and $\|a_i^*\| \leq N$ for every $i \leq k$, where \bar{a}^* denotes the dual basis of \bar{a} . We denote by $\mathcal{E}_1(k, N)$ the set of *N-Auerbach* $\bar{a} \in \mathcal{E}_1(k)$. Observe that in view of Corollary 3.3 the set $\mathcal{E}_1(k, N)$ is closed in $\mathcal{E}_1(k)$. Moreover it can be easily verified that if $\bar{a} \in \mathcal{E}_1(k, N)$ and $\alpha_i \in M_n$ then

$$\left\| \sum_i \alpha_i \otimes a_i \right\| \leq N \left\| \sum_i \alpha_i \otimes e_i \right\|$$

and

$$\left\| \sum_i \alpha_i \otimes e_i \right\| \leq kN \left\| \sum_i \alpha_i \otimes e_i \right\|.$$

If $\bar{a}, \bar{b} \in \mathcal{E}_1(k)$, denote as in Subsection 3.2 by $\iota_{\bar{a}, \bar{b}}$ the linear isomorphism from $\langle \bar{a} \rangle$ to $\langle \bar{b} \rangle$ such that $\iota_{\bar{a}, \bar{b}}(a_i) = b_i$ for $i \leq k$. Define the completely bounded distance $d_{cb}(\bar{a}, \bar{b})$ to be $\|\iota_{\bar{a}, \bar{b}}\|_{cb} \|\iota_{\bar{a}, \bar{b}}^{-1}\|_{cb}$.

Proposition 4.5. Suppose that $\bar{a}, \bar{b} \in \mathcal{E}_1(k, N)$. Then

$$d_{cb}(\bar{a}, \bar{b}) \leq (1 + kNd_{\mathcal{E}_1}(\bar{a}, \bar{b}))^2$$

and

$$d_{\mathcal{E}_1}(\bar{a}, \bar{b}) \leq d_{cb}(\bar{a}, \bar{b}) - 1.$$

Proof. The first inequality can be inferred from Proposition 4.3; see also the proof of the first inequality in Proposition 3.4. The second inequality is an immediate consequence of Proposition 4.2. \square

Using Proposition 4.5 one can show that $(\mathcal{E}_1(k), d_{\mathcal{E}_1})$ is a separable metric space. The proof is similar to [28, Proposition 12]. Recall that $\mathcal{M}_\infty^0 \subset \mathcal{E}_1$ denotes the class of operator spaces that admit a completely isometric embedding into M_n for some $n \in \mathbb{N}$.

Proposition 4.6. For every $k \in \mathbb{N}$, $(\mathcal{E}_1(k), d_{\mathcal{E}_1})$ is a complete metric space, and $\mathcal{M}_\infty^0(k)$ is a dense subset of $\mathcal{E}_1(k)$.

Proof. Suppose that $(\bar{a}^{(m)})_{m \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{E}_1(k)$. By Proposition 4.8 there is $N \in \mathbb{N}$ such that, for every $m \in \mathbb{N}$, $\bar{a}^{(m)} \in \mathcal{E}_1(k, N)$. Moreover

$$\limsup_{k, m \rightarrow +\infty} d_{cb}(\bar{a}^{(m)}, \bar{a}^{(k)}) = 1$$

Fix a nonprincipal ultrafilter \mathcal{U} over \mathbb{N} . Define X to be the ultraproduct $\prod_{\mathcal{U}} \langle \bar{a}^{(m)} \rangle$ as in [8, Section 10.3]. Let a_i be the element of X having $(a_i^{(m)})_{m \in \mathbb{N}}$ as representative sequence. Observe that for every $n, m \in \mathbb{N}$ and

$\alpha_i \in M_n$,

$$\begin{aligned}
& \left\| \left\| \sum_i \alpha_i \otimes a_i^{(m)} \right\| - \left\| \sum_i \alpha_i \otimes a_i \right\| \right\| \\
&= \lim_{k \rightarrow \mathcal{U}} \left\| \left\| \sum_i \alpha_i \otimes a_i^{(m)} \right\| - \left\| \sum_i \alpha_i \otimes a_i^{(k)} \right\| \right\| \\
&\leq \sup_{k \geq m} d_{cb}(\bar{a}^{(m)}, \bar{a}^{(k)}).
\end{aligned}$$

Therefore $d_{cb}(\bar{a}^{(m)}, \bar{a}) \rightarrow 1$. Hence again by Proposition 4.8 we have that $d_{\mathcal{E}_1}(\bar{a}^{(m)}, \bar{a}) \rightarrow 0$. This shows that \bar{a} is the limit of the Cauchy sequence $(\bar{a}^{(m)})_{m \in \mathbb{N}}$. Observe now that by Proposition 4.8 and by the definition of 1-exact operator space, $\mathcal{M}_\infty^0(k)$ is dense in $(\mathcal{E}_1(k), d_{\mathcal{E}_1})$. It follows from Proposition 3.5 that $\mathcal{M}_\infty^0(k)$ is a separable subspace of $\mathcal{E}_1(k)$. Alternatively one can observe that the algebra \mathcal{K} of compact operators contains all elements of $\mathcal{M}_\infty^0(k)$, and then apply Corollary [29, Corollary 2.13.3] and Proposition 4.5. In any case one can conclude that $\mathcal{E}_1(k)$ is separable as well. \square

4.5. The noncommutative Gurarij space as a Fraïssé limit . We have shown in Subsection 4.3 that \mathcal{E}_1 is a Fraïssé class. Therefore we can consider its corresponding Fraïssé limit according to Theorem 2.9. First we observe that using Proposition 4.3 one can reformulate the property of being limit.

The following notion has been introduced by Oikhberg in [25].

Definition 4.7. An operator space Z is *noncommutative Gurarij* if for any finite-dimensional 1-exact operator systems $X \subset Y$, complete isometry $\phi : X \rightarrow Z$, and $\varepsilon > 0$, there is an injective linear map $\psi : Y \rightarrow Z$ extending ϕ such that $\|\psi\|_{cb} \|\psi^{-1}\|_{cb} < 1 + \varepsilon$.

The following proposition characterizes noncommutative Gurarij spaces as limits of the Fraïssé class \mathcal{E}_1 .

Proposition 4.8. Suppose that Z is a separable 1-exact. The following conditions are equivalent:

- (1) Z is a Fraïssé limit of the class \mathcal{E}_1 ;
- (2) For every $n \in \mathbb{N}$, subspace X of M_n , complete isometry $\phi : X \rightarrow Z$, and $\varepsilon > 0$ there is a complete isometry $\psi : M_n \rightarrow Z$ such that $\|\psi|_X - \phi\|_{cb} < \varepsilon$;
- (3) Z is noncommutative Gurarij;
- (4) Z satisfies the universal property of Definition 4.7 when $Y = M_n$ for some $n \in \mathbb{N}$;
- (5) For every $n, k, l \in \mathbb{N}$, for every subspace X of M_n , finite l -tuple \bar{b} in X , $\psi \in \text{Stx}_{\mathcal{E}_1}^{<\psi}(X, Z)$, and $\varepsilon > 0$, there is $\varphi \in \text{Stx}_{\mathcal{E}_1}^{<\psi}(X, Z)$ such that for every $i \leq l$ there is $y \in Z$ such that $\varphi(b_i, y) < \varepsilon$.

Proof. The implications (1) \Rightarrow (2) and (3) \Rightarrow (4) are obvious. The implications (2) \Rightarrow (4) and (1) \Rightarrow (3) can be proved using Proposition 4.5 similarly as the

implication (1) \Rightarrow (2) in Proposition 3.6. The implication (4) \Rightarrow (5) can be proved as (2) \Rightarrow (1) of Proposition 3.6, or [1, Theorem 3.3]. Finally (5) \Rightarrow (1) is a consequence of [1, Lemma 2.16] and the fact that $\mathcal{M}_\infty^0(k)$ is dense in $(\mathcal{E}_1(k), d_{\mathcal{E}_1})$; see Proposition 4.6. \square

With such a characterization of a limit of the Fraïssé class \mathcal{E}_1 at hand, we can finally state the main result of this paper, which is an immediate consequence of Theorem 2.9 and Proposition 4.8.

Theorem 4.9. There is a separable 1-exact operator space $\mathbb{N}\mathbb{G}$ which is noncommutative Gurarij. Such a space is unique up to completely isometric isomorphism. Every separable 1-exact operator space can be completely isometrically embedded into $\mathbb{N}\mathbb{G}$. Moreover $\mathbb{N}\mathbb{G}$ has the following homogeneity property: If $X \subset \mathbb{N}\mathbb{G}$ is finite-dimensional, $\phi : X \rightarrow \mathbb{N}\mathbb{G}$ is a complete isometry, and $\varepsilon > 0$, then there is a completely isometric surjection $\psi : \mathbb{N}\mathbb{G} \rightarrow \mathbb{N}\mathbb{G}$ such that $\|\psi|_X - \phi\|_{cb} < \varepsilon$.

Theorem 1.1 of [25] shows that any two noncommutative Gurarij spaces are *approximately* completely isometric. The uniqueness assertion in Theorem 4.9 improves such a result, showing that any two noncommutative Gurarij spaces are (exactly) completely isometric. Assuming uniqueness, one can also deduce universality from [25, Theorem 1.1] together with the fact that every separable 1-exact operator space embeds into a separable $\mathcal{OL}_{\infty,1+}$ space; see [6, Theorem 4.7].

Recall that an operator space X is an $\mathcal{OL}_{\infty,1+}$ space as defined in [7] if for every finite-dimensional subspace E of X and every $\varepsilon > 0$ there is a finite-dimensional C^* -algebra A and a subspace F of A such that $d_{cb}(E, F) < 1 + \varepsilon$. This notion provides the noncommutative analog of $\mathcal{L}_{\infty,1+}$ spaces as in [22]. Clearly $\mathcal{OL}_{\infty,1+}$ spaces are closed under direct limits. Therefore from Remark 2.10 and Proposition 4.8 one can deduce the following fact, already observed by Oikhberg in [25].

Proposition 4.10. The noncommutative Gurarij space is an $\mathcal{OL}_{\infty,1+}$ space.

We now want to observe that there is a tight connection between the noncommutative Gurarij space $\mathbb{N}\mathbb{G}$ and the Gurarij Banach space \mathbb{G} . More generally the same applies to the M_n -spaces \mathbb{G}_n defined in Subsection 3.3. It should be noted that the spaces \mathbb{G}_n are pairwise not completely isometric and, in turn, not completely isometric to $\mathbb{N}\mathbb{G}$. In fact on one hand M_{n+1} embeds completely isometrically into \mathbb{G}_{n+1} and $\mathbb{N}\mathbb{G}$ by universality. On the other hand there is no complete isometry from M_{n+1} to \mathbb{G}_n since M_{n+1} is not an M_n -space. This can be deduced from the fact that the transposition map θ is a linear isometry of M_{n+1} such that $\|\theta\|_{n+1} = n+1$ by [32, Theorem 1.2].

Proposition 4.11. The noncommutative Gurarij space $\mathbb{N}\mathbb{G}$ is n -isometrically isomorphic to \mathbb{G}_n . In particular $\mathbb{N}\mathbb{G}$ is isometrically isomorphic to the Gurarij Banach space.

Proof. Consider the M_n -space $\text{MIN}_n(\text{NG})$. Observe that by the properties of the MIN and MAX constructions, $\text{MIN}_n(\text{NG})$ satisfies the equivalent conditions of Proposition 3.6. Therefore by the uniqueness assertion in Theorem 3.7, $\text{MIN}_n(\text{NG})$ is n -isometrically isomorphic to \mathbb{G}_n . \square

By Theorem 4.9 and Proposition 4.11 the noncommutative Gurarij space NG can be regarded as a canonical operator space structure on the Gurarij Banach space.

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